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EQUIVALENT CONDITIONS TO THE SPECTRAL DECOMPOSITION PROPERTY FOR CLOSED OPERATORS

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ABSTRACT. The spectral decomposition property has been instrumental in developing a local spectral theory for closed operators acting on a complex Banach space. This paper gives some necessary and sufficient conditions for a closed operator to possess the spectral decomposition property.

In the monograph [3] and in a sequel of papers by the authors, a local spectral theory has been built for closed operators on the sole assumption of the spectral decomposition property. As an abstraction of Dunford's concept of "spectral reduction' [2, p. 1927] and that of Bishop's "duality theory of type 3" [1], an operator T endowed with the spectral decomposition property produces a spectral decomposition of the underlying space, pertinent to any finite open cover of the spectrum $\sigma(T)$. In this paper we obtain some conditions equivalent to the spectral decomposition property. Some of them generalize results from [4].

1. PRELIMINARIES

Given a Banach space X over the complex field $\mathbf{C}$, we denote by $C(X)$ the class of closed operators with domain D_T and range in X, and we write $C_d(X)$ for the subclass of the densely defined operators in $C(X)$. For a subset Y of X, $Y^\perp$ denotes the annihilator of Y in X^* and for $Z \subset X^*$, we use the symbol ${}^\perp Z$ for the preannihilator of Z in X. For the rest, the terminology and notation conform to that employed in [3].

We shall adopt and adjust some concepts and ideas from Bishop's "duality theory of type 4" [1, §4]. A couple U_1 and U_2 of a bounded and an unbounded Cauchy domain, related by $U_2 = (\overline{U}_1)^c$, are referred to as complementary simple sets. W_1 and W_2 are the sets of analytic functions from U_1 to X and from U_2 and X^*, respectively, which vanish at ∞. The seminorms

$$\|f\|_{K_1} = \max\{\|f(\lambda)\|: f \in W_1, \lambda \in K_1, K_1\ (\subset U_1) \text{ is compact}\}$$

and

$$\|g\|_{K_2} = \max\{\|g(\lambda)\|: g \in W_2, \lambda \in K_2, K_2\ (\subset U_2) \text{ is compact}\}$$

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induce a locally convex topology on W_1 and W_2, respectively. For $i = 1,2$ let V_i be the subset of W_i on which every function can be extended to be continuous on $\overline{U}_i$. For $f \in V_1$, $g \in V_2$, the norms

$$\|f\|_{V_1} = \sup\{\|f(\lambda)\| : \lambda \in U_1\}, \quad \|g\|_{V_2} = \sup\{\|g(\lambda)\| : \lambda \in U_2\}$$

make $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ Banach spaces. For $x \in X$, $\mu \in U_1$ and $\lambda \in U_2$, define

$$\alpha(x, \mu, \lambda) = (\mu - \lambda)^{-1} x. \tag{1.1}$$

For fixed $x \in X$ and $\lambda \in U_2$, $\alpha(x, \cdot, \lambda)$ is called an elementary element of V_1. Denote by V the subspace of V_1 which is spanned by the elementary elements of V_1. For $x^* \in X^*$, $\mu \in U_1$ and $\lambda \in U_2$, define

$$\alpha(x^*, \mu, \lambda) = (\mu - \lambda)^{-1} x^*. \tag{1.2}$$

For fixed $x^* \in X^*$ and $\mu \in U_1$, call $\alpha(x^*, \mu, \cdot)$ and elementary element of V_2. For $f \in V_1$ and $g \in V_2$, with continuous extensions to $\Gamma = \partial U_1 = \partial U_2$, endow Γ with the clockwise orientation and ascertain that the bilinear functional

$$\Phi(f) = \langle f, g \rangle = \frac{1}{2\pi i} \int_\Gamma \langle f(\lambda), g(\lambda) \rangle \, d\lambda \tag{1.3}$$

is jointly continuous.

For $g \in V_2$, (1.3) defines a bounded linear functional Φ on $\dot{V}$, i.e. $\Phi \in V^*$. For $f = \alpha(x, \cdot, \lambda)$, one obtains

$$\Phi(f) = \frac{1}{2\pi i} \int_\Gamma (\mu - \lambda)^{-1} \langle x, g(\mu) \rangle \, d\mu = \langle x, g(\lambda) \rangle. \tag{1.4}$$

The last equality holds because $g(\infty) = 0$.

The proof of the following lemma is similar to that of [3, Lemma 9.1].

1.1. Lemma. *Let U_1, U_2 be complementary simple sets. With V, V_i and W_i $(i = 1,2)$, as defined above, there exists a linear manifold Y in W_2 and a norm on Y such that*

(i) *Y is a Banach space isometrically isomorphic to V^*:*
(ii) *$V_2 \subset Y$;*
(iii) *the mappings $V_2 \to Y$ and $Y \to W_2$ are continuous;*
(iv) *the inner product between V and V_2, defined by* (1.3), *can be extended to an inner product between V and Y in conjunction with the isometric isomorphism between Y and V^*, as asserted by* (i).

2. Some dual properties

For an operator $T \in C_d(X)$, define an operator H on V by

$$D_H = \{f \in V : Tf(\mu) \in V\}, \quad (Hf)(\mu) = (\mu - T)f(\mu).$$

2.1. Lemma. *The operator H is closed and densely defined on V.*

Proof. For $f = \alpha(x, \cdot, \lambda)$ with $x \in D_T$, one has $f \in D_H$ and

$$(Hf)(\mu) = (\mu - \lambda)^{-1}(\mu - T)x = x + (\mu - \lambda)^{-1}(\lambda - T)x. \tag{2.1}$$

The linear span of all elementary elements being dense in V, the operator H is densely defined.

Let $\{f_n\}$ be a sequence in D_H such that $f_n \to f$ and $Hf_n \to g$, for some functions f and g. T being closed, it follows from

$$(Hf_n)(\mu) = (\mu - T)f_n(\mu),$$

that $f \in D_H$ and

$$(Hf)(\mu) = (\mu - T)f(\mu) = g(\mu), \qquad \mu \in \overline{U}_1.$$

Thus H is closed. □

The next lemma defines the dual H^* of H. Henceforth, g will denote a typical element of $V^* = Y$.

2.2. Lemma. *The dual operator of H is defined by*

$$(H^* g)(\lambda) = -\lim_{\lambda\to\infty} \lambda g(\lambda) + (\lambda - T^*)g(\lambda), \qquad g \in D_{H^*}. \tag{2.2}$$

Proof. For $f(\mu) = \alpha(x, \mu, \lambda)$ with $x \in D_T$ and $\lambda \in U_2$ fixed, and $g \in D_{H^*}$, (2.1), (1.3) and (1.4) imply

$$\begin{aligned} \langle f, H^* g\rangle &= \langle Hf, g\rangle = \langle x, g\rangle + \langle \alpha((\lambda - T)x, \cdot, \lambda), g\rangle \\ &= \langle x, g\rangle + \langle (\lambda - T)x, g(\lambda)\rangle, \end{aligned} \tag{2.3}$$

where x, as a function of $\mu \in \overline{U}_1$, is an element of V_1. It follows from

$$x = \frac{1}{2\pi i}\int_{\Gamma'} (\mu - \lambda)^{-1} x \, d\lambda = \frac{1}{2\pi i}\int_{\Gamma'} \alpha(x, \mu, \lambda)\, d\lambda,$$

that $x \in V$, where Γ' is a closed C^2-Jordan curve with the clockwise orientation that contains Γ in its interior. Furthermore, with the help of (1.4), one obtains

$$\begin{aligned} \langle x, g\rangle &= \frac{1}{2\pi i}\int_{\Gamma'} \langle \alpha(x, \cdot, \lambda), g\rangle \, d\lambda = \frac{1}{2\pi i}\int_{\Gamma'} \langle x, g(\lambda)\rangle \, d\lambda \\ &= \lim_{z\to\infty} z\left(-\frac{1}{2\pi i}\int_{\Gamma'} (\lambda - z)^{-1}\langle x, g(\lambda)\rangle \, d\lambda\right) = -\lim_{z\to\infty} z\langle x, g(z)\rangle \\ &= -\lim_{\lambda\to\infty} \lambda\langle x, g(\lambda)\rangle = \left\langle x, -\lim_{\lambda\to\infty} \lambda g(\lambda)\right\rangle. \end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned} \langle f, H^* g\rangle &= \left\langle x, -\lim_{\lambda\to\infty} \lambda g(\lambda)\right\rangle + \langle (\lambda - T)x, g(\lambda)\rangle \\ &= \left\langle x, -\lim_{\lambda\to\infty} \lambda g(\lambda) + (\lambda - T)^* g(\lambda)\right\rangle. \end{aligned} \tag{2.5}$$

In fact, $f = \alpha(x,\cdot,\lambda)$ and since $\langle f, H^*g\rangle$ and $\langle x, -\lim_{\lambda\to\infty}\lambda g(\lambda)\rangle$ are bounded linear functionals of x, so is $\langle(\lambda - T)x, g(\lambda)\rangle$. Thus $g(\lambda) \in D_{T^*}$, for every $\lambda \in U_2$ and hence the last equality of (2.5) holds. Now (2.5) combined with (1.3) and (1.4), gives $\langle f, H^*g\rangle = \langle x, (H^*g)(\lambda)\rangle$ and hence H^* is expressed by (2.2). □

Define the map $\tau\colon V^* \to X^*$ by $\tau g = \lim_{\lambda\to\infty}\lambda g(\lambda)$. Then H^*, given by (2.2), is expressed by

$$(H^*g)(\lambda) = -\tau g + (\lambda - T^*)g(\lambda). \tag{2.6}$$

2.3. Lemma. *Let $x^* \in X^*$. Then $x^* \in D_{T^*}$ iff there exists $g \in D_{H^*}$ such that $\tau g = x^*$.*

Proof. First, assume that there is $g \in D_{H^*}$ such that $\tau g = x^*$. Since $H^*g \in V^*$, the following limit exists

$$\lim_{\lambda\to\infty} T^*\lambda g(\lambda) = \lim_{\lambda\to\infty} \lambda T^* g(\lambda).$$

Furthermore, $\lim_{\lambda\to\infty}\lambda g(\lambda)$ also exists and since T is closed, we have

$$x^* = \tau g = \lim_{\lambda\to\infty}\lambda g(\lambda) \in D_{T^*}.$$

Conversely, for every $x^* \in D_{T^*}$, the corresponding elementary element $\alpha(x^*, \mu, \cdot)$ with $\mu \in U_1$ fixed, is in D_{H^*}. It follows from (1.2), that

$$\tau(-\alpha) = -\lim \lambda\alpha(x^*, \mu, \lambda) = x^*$$

and the proof reaches its conclusion by setting $g = -\alpha$. □

3. Norms on the dual spaces

We introduce the norm

$$\|(f_1, f_2)\|_\eta = (\eta\|f_1\|^2 + \|f_2\|^2)^{1/2}, \quad \eta > 0,$$

in $V \oplus V$. This induces the norm

$$\|(g_1, g_2)\|_\eta = (\eta^{-1}\|g_1\|^2 + \|g_2\|^2)^{1/2}$$

in $V^* \oplus V^*$. Let $G(H)$ and $G(H^*)$ be the graphs of H and H^*, respectively. $G(H)$, as a subspace of $V \oplus V$, is endowed with the norm

$$\|(f, Hf)\|_\eta = (\eta\|f\|^2 + \|Hf\|^2)^{1/2}. \tag{3.1}$$

It follows from

$$(G(H))^\perp = \nu G(H^*), \quad \text{where } \nu(g_1, g_2) = (-g_2, g_1),$$

that $\nu G(H^*)$ is the dual of $(V \oplus V)/G(H)$. The latter is equipped with the norm

$$\|(f_1, f_2)^\frown\|_\eta = \inf\{(\|f_1 - f\|^2 + \|f_2 - Hf\|^2)^{1/2} \colon f \in D_H\}, \tag{3.2}$$

where $(f_1, f_2)^\frown$ denotes a typical element of $(V \oplus V)/G(H)$. To the norm (3.2), there corresponds the following norm in $\nu G(H^*)$:

$$\|(-H^* g, g)\|_\eta = (\eta^{-1}\|H^* g\|^2 + \|g\|^2)^{1/2}.$$

3.1. Lemma. *The norm*

$$\|x^*\|_{T^*} = (\|x^*\|^2 + \|Tx^*\|^2)^{1/2} \tag{3.3}$$

in D_{T^*} is equivalent to the norm

$$\|x^*\|_\eta = \inf\{(\eta^{-1}\|H^* g\|^2 + \|g\|^2)^{1/2} : \tau g = x^*\}. \tag{3.4}$$

Furthermore, D_{T^*} equipped with the norm (3.3) *or* (3.4) *is the dual of a Banach space.*

Proof. First, we prove that D_{T^*} endowed with the norm (3.4) is a Banach space. Let $\{x_n^*\}$ be a Cauchy sequence with respect to the norm (3.4). Without loss of generality, we may suppose that

$$\sum_{n=0}^{\infty} \|x_{n+1}^* - x_n^*\|_\eta < \infty, \qquad x_0 = 0. \tag{3.5}$$

For each x_n, we may choose $g_n \in D_{H^*}$ such that

$$(\eta^{-1}\|H^*(g_{n+1} - g_n)\|^2 + \|g_{n+1} - g_n)\|^2)^{1/2} \le 2\|x_{n+1}^* - x_n^*\|_\eta \tag{3.6}$$

and $\tau g_n = x_n^*$. Relations (3.5) and (3.6) imply that both $\{g_n\}$ and $\{H^* g_n\}$ converge. Put $g = \lim_{n\to\infty} g_n$. H^* being closed, one has $g \in D_{H^*}$ and $H^* g = \lim_{n\to\infty} H^* g_n$. Then Lemma 2.3 implies that $x^* = \tau g \in D_{T^*}$. Since

$$\|x_n^* - x^*\|_\eta \le (\eta^{-1}\|H^*(g_n - g)\|^2 + \|g_n - g\|^2)^{1/2} \to 0 \quad n \to \infty,$$

it follows that D_{T^*}, endowed with the norm (3.4), is a Banach space.

To show that the norms (3.3) and (3.4) are equivalent, let $x^* \in D_{T^*}$ and $g = \alpha(x^*, \mu, \cdot)$ with $\mu \in U_1$ fixed. Since $\tau g = x^*$, one has

$$\begin{aligned} \|x^*\|_\eta &\le (\eta^{-1}\|H^* g\|^2 + \|g\|^2)^{1/2} \\ &\le \frac{1}{\delta}(\eta^{-1}(|\mu| \cdot \|x^*\| + \|T^* x^*\|)^2 + \|x^*\|^2)^{1/2}, \end{aligned} \tag{3.7}$$

where $\delta = \operatorname{dist}(\mu, U_2)$. In view of (3.7), there exists $K_\eta > 0$ such that

$$\|x^*\|_\eta \le K_\eta(\|x^*\|^2 + \|T^* x^*\|^2)^{1/2} = K_\eta \|x^*\|_{T^*}. \tag{3.8}$$

D_{T^*} being complete with respect to both $\|\cdot\|_\eta$ and $\|\cdot\|_{T^*}$, (3.8) implies that the two norms are equivalent. D_{T^*} equipped with the norm (3.3) is isometrically isomorphic to $\nu G(T^*)$ $(= G(T)^\perp)$. Since $\nu G(T^*)$ is the dual of $X \oplus X/G(T)$, so is D_{T^*}. □

D_{T^*} equipped with either of the two norms (3.3), (3.4), will be denoted by D. To obtain a further property of τ, we need the following.

3.2. Lemma. *Let Y, Z be Banach spaces and let S be a bounded surjective map of Y onto Z. In Z we define the norm*

$$\|z\|_S = \inf\{\|y\| : y \in Y, Sy = z\}, \qquad z \in Z. \tag{3.9}$$

Then, the corresponding norm in the dual space Z^* is

$$\|z^*\|_{S^*} = \|S^* z^*\|, \qquad z^* \in Z^*. \tag{3.10}$$

Proof. Let $N(S)$ be the null space of S. Then $N(S)^\perp$ is the dual of $Y/N(S)$. Let $y_0 \in Y$, $z = Sy_0$ and let $\hat{y}_0$ be the equivalence class of y_0 in $Y/N(S)$. In terms of the norm (3.9), one has

$$\|\hat{y}_0\| = \inf\{\|y_0 - w\| : w \in N(S)\} = \inf\{\|y\| : Sy = z\} = \|z\|_S.$$

The dual norm of $\|\hat{y}_0\|$ in $N(S)^\perp$ is the usual norm in Y^*, restricted to $N(S)^\perp$. Note that S^* is a surjective map from Z^* onto $N(S)^\perp$. Therefore, the corresponding norm of $\|\cdot\|_S$ in Z^* is the one expressed by (3.10). □

We define an operator K from $\nu G(H^*)$ into D_{T^*} by $K(-H^* g, g) = \tau g$. Let $D^\#$ be the dual of D. Then $K^\#$, the dual of K, is an operator from $D^\#$ into the dual of $\nu G(H^*)$, i.e. from $D^\#$ into $(V^{**} \oplus V^{**})/(\nu G(H^*))^\perp$.

For every $x \in X$, define a continuous linear functional ψ on D, by

$$\psi(x^*) = \langle x, x^* \rangle, \qquad x^* \in D. \tag{3.11}$$

3.3. Lemma. *The linear functional ψ (3.11) is a zero functional only if $x = 0$.*

Proof. Assume that $\psi = 0$. Then $\langle x, x^* \rangle = 0$ for every $x^* \in D$. Thus, we have $\langle 0, -T^* x^* \rangle + \langle x, x^* \rangle = 0$, $x^* \in D$, equivalently, $(0, x) \perp \nu G(T^*)$, i.e. $(0, x) \in G(T)$. Consequently, $x = 0$. □

In view of Lemma 3.3, we may consider X as a subset of $D^\#$. In the following, we shall have a closer look at $K^\# x$ for $x \in X$.

For $x^* \in D$ and fixed $\mu \in U_1$, put $g = \alpha(x^*, \mu, \cdot)$. Then, one obtains

$$\begin{aligned} \langle K^\# x(-H^* g, g) \rangle &= \langle x, K(-H^* g, g) \rangle = \langle x, \tau g \rangle = \langle x, x^* \rangle \\ &= \frac{1}{2\pi i} \int_\Gamma (\mu - \lambda)^{-1} \langle x, x^* \rangle \, d\lambda = \langle x, g \rangle, \end{aligned}$$

where Γ has a clockwise orientation.

We know that $(0, x)$ is an element of $X \oplus X$. We may also consider $(0, x)$ as an element of $V \oplus V$ and hence $(0, x)$ can be assumed to be an element of $V^{**} \oplus V^{**}$.

Thus, we have

$$\langle x, g \rangle = \langle (0, x), (-H^* g, g) \rangle = \langle (0, x)^\sim, (-H^* g, g) \rangle,$$

where $(0, x)^\sim$ is the equivalence class of $(0, x)$ in $(V^{**} \oplus V^{**})/(\nu G(H^*))^\perp$. Consequently, $K^\# x = (0, x)^\sim$.

Denote by $(0, x)^\wedge$ the equivalence class of $(0, x)$ in $(V \oplus V)/G(H)$.

3.4. Lemma. *Let* J *be the natural embedding of* $(V \oplus V)/G(H)$ *into*

$$(V^{**} \oplus V^{**})/(\nu G(H^*))^\perp.$$

Then $J(0,x)\hat{} = (0,x)\tilde{}$.

Proof. For any $(-H^*g, g) \in \nu G(H^*)$, one has

$$\begin{aligned}\langle (0,x)\hat{}, (-H^*g, g)\rangle &= \langle (0,x), (-H^*g, g)\rangle = \langle (-H^*g, g), (0,x)\rangle \\ &= \langle (-(H^*g, g), (0,x)\tilde{}\rangle.\end{aligned}$$

Note that while in $\langle (0,x), (-H^*g, g)\rangle$, $(0,x) \in V \oplus V$; in $\langle (-H^*g, g), (0,x)\rangle$, $(0,x)$ is considered an element of $V^{**} \oplus V^{**}$. It follows from the above equalities that $J(0,x)\hat{} = (0,x)\tilde{}$. □

In particular, Lemma 3.4 implies

$$\|(0,x)\hat{}\| = \|(0,x)\tilde{}\|. \tag{3.12}$$

On the other hand, $(0,x)\hat{} = 0$ implies $(0,x) \in G(H)$ and the latter implies $x = 0$. Accordingly, we may define the following norm on X:

$$\|x\|_\eta = \|(0,x)\hat{}\| = \inf\{(\eta\|f\|^2 + \|x - Hf\|^2)^{1/2} : f \in D_H\}. \tag{3.13}$$

In view of Lemma 3.4, we may consider $K^\# x = (0,x)\tilde{}$ as a point of $(V \oplus V)/G(H)$.

3.5. Lemma. *The norm* $\|\cdot\|_\eta$, *defined by* (3.13), *is the restriction of the norm on* $D^\#$.

Proof. The space $(V^{**} \oplus V^{**})/(\nu G(H^*))^\perp$ is the conjugate of $\nu G(H^*)$ and K is an operator from $\nu G(H^*)$ into D. It follows from Lemma 3.2, that the dual norm on $D^\#$ is given by

$$\|x^\#\| = \|K^\# x^\#\|_{**} \tag{3.14}$$

where $x^\# \in D^\#$ and $\|\cdot\|_{**}$ is the norm on $V^{**} \oplus V^{**}/[\nu G(H^*)]^\perp$. If $x^\# = x \in X$, then the norm (3.14) becomes $\|x\|_\eta = \|K^\# x\| = \|(0,x)\tilde{}\|$ and it follows from (3.12) that the restriction of the norm (3.14) to X is that given by (3.13). □

4. A DUALITY PROPERTY OF SOME SPECTRAL-TYPE MANIFOLDS

Define the following linear manifolds in X:

$$N = \{x \in X\colon \text{for every } \varepsilon > 0,\ \text{there exists } f \in D_H \text{ with } \|x - Hf\| < \varepsilon\},$$

$$M = \{x^* \in D\colon \text{there exists } g \in D_{H^*} \text{ such that } H^* g = 0, \tau g = x^*\}.$$

4.1. Lemma. *The manifolds* N *and* M *have the following characterizations*:

$$N = \{x \in X\colon \|x\|_\eta \to 0 \textit{ as } \eta \to 0\}, \tag{4.1}$$

$$M = \{x^* \in D\colon \|x^*\|_\eta \le R \textit{ for } n > 0 \textit{ and } R \textit{ depends on } x^*\}. \tag{4.2}$$

Proof. First, we establish (4.1). Let $x \in N$. Since, for every $\varepsilon > 0$, there is $f \in D_H$ such that $\|x - Hf\| < \varepsilon$, it follows from (3.13) that $\overline{\lim}_{\eta\to 0} \|x\|_\eta \le \varepsilon$. ε being arbitrary, it follows that $\lim_{\eta\to 0} \|x\|_\eta = 0$.

Conversely, suppose that $\|x\|_\eta \to 0$ as $\eta \to 0$. Then, for every $\varepsilon > 0$, there exists $\eta > 0$ such that $\|x\|_\eta < \varepsilon$ and hence $\|x - Hf\| < \varepsilon$ for some $f \in D_H$.

Next, we prove (4.2). It is a straightforward consequence of (3.4) that

$$M \subset \{x^* \in D: \|x^*\|_\eta \text{ is bounded for } \eta > 0\}.$$

Conversely, suppose that $x^* \in D$ and $\|x^*\|_\eta$ is bounded for $\eta > 0$, i.e. there exists $R > 0$ such that

$$\inf\{(n\|H^* g\|^2 + \|g\|^2)^{1/2}, \tau g = x^*\} < R, \quad n = 1, 2, \dots.$$

Then, for every n, there exists $g_n \in D_{H^*}$ satisfying conditions

$$n\|H^* g_n\|^2 + \|g_n\|^2 \le R^2 \quad \text{and} \quad \tau g_n = x^*. \tag{4.3}$$

In view of (4.3), the sequences $\{g_n\}$ and $\{H^* g_n\}$ are bounded and hence the sequence $\{(H^* g_n, g_n)\}$ is bounded. Consequently, $\{(-H^* g_n, g_n)\}$ has a cluster point (h, g) in $V^* \oplus V^*$, with respect to the weak* topology of $V^* \oplus V^*$. Since $\nu G(H^*)$ is closed with respect to the same topology, one has $\{h, g\} \in \nu G(H^*)$, i.e. $h = -H^* g$. It follows from $\|H^* g_n\| \le R^2/n$ that $\|H^* g\| = 0$.

On the other hand, for every $x \in X$, $K^\# x = (0, x)^\sim \in (V \oplus V)/G(H)$. Therefore,

$$\langle x, x^* \rangle = \langle x, \tau g_n \rangle = \langle x, K(-H^* g_n, g_n) \rangle = \langle K^\# x, (-H^* g_n, g_n) \rangle.$$

Since $(-H^* g, g)$ is also a cluster point of $\{(-H^* g_n, g_n)\}$ in the weak* topology of $\nu G(H^*)$, the latter being the dual space of $(V \oplus V)/G(H)$, we have

$$\langle x, x^* \rangle = \langle K^\# x, (-H^* g, g) \rangle = \langle x, K(-H^* g, g) \rangle = \langle x, \tau g \rangle.$$

Thus $\tau g = x^*$ and hence $x^* \in M$. Expression (4.2) is obtained. □

4.2. Theorem. *N and M, as defined above, are related by*

$$N^\perp = \overline{M}^w,$$

where ^{w} *denotes the weak* closure in* X^*.

Proof. Let $x \in N$ and $x^* \in M$. It follows from Lemmas 3.5 and 4.1, that

$$|\langle x, x^* \rangle| \le \|x\|_\eta \cdot \|x^*\|_\eta \to 0 \quad (\text{as } \eta \to 0).$$

Therefore, $N^\perp \supset \overline{M}^w$.

Next, we prove the opposite inclusion. For $x \notin N$ $(x \in X)$, Lemma 4.1 implies that there exists $\eta_n \downarrow 0$ such that

$$\|x\|_{\eta_n} > C > 0 \tag{4.4}$$

for some constant C. In view of (4.4), we can find $x_n^* \in D$ such that $\|x_n^*\|_{\eta_n} \le 1$ and $|\langle x, x_n^* \rangle| > C$. The sequence $\{\eta_n\}$ being nonincreasing, so is the norm

(3.4), i.e. $\|x_n^*\|_{\eta_1} \le \|x_n^*\|_{\eta_n}$. Consequently, $\{x_n^*\}$ is bounded in the norm $\|\cdot\|_{\eta_1}$-topology. For every n, there exists $g_n \in D_{H^*}$ such that

$$\eta_n^{-1}\|H^* g_n\|^2 + \|g_n\|^2 \le 2\|x_n^*\|_{\eta_n}^2.$$

Thus $\{(x_n^*, g_n, H^* g_n)\}$ is bounded in $D \oplus G(H^*)$. By Lemma 3.1 and the previous paragraph, $D\oplus G(H^*)$ is the dual of a Banach space and $\{(x_n^*, g_n, H^* g_n)\}$ has a cluster point $(x^*, g, H^* g)$ in the weak* topology of $D \oplus G(H^*)$. Since (3.11) defines a continuous linear functional on D for every $x \in X$, it follows that x^* is also a cluster point of $\{x_n^*\}$ in the weak* topology of X^*. Now it follows from the inequalities

$$\langle x, x_n^*\rangle = \langle x, \tau g_n\rangle = \langle x, K(-H^* g_n, g_n)\rangle = \langle K^\# x, (-H^* g_n, g_n)\rangle$$

that, for $x \in X$, one has

$$\langle x, x^*\rangle = \langle K^\# x, (-H^* g, g)\rangle = \langle x, K(-H^* g, g)\rangle = \langle x, \tau g\rangle.$$

Thus $x^* = \tau g$. By the definition of M, $x^* \in M$. Hence $N \supset^\perp M$, or equivalently, $N^\perp \subset \overline{M}^w$. □

5. The main theorem

We recall the definition of the central topic of this paper.

5.1. Definition. An operator $T \in C(X)$ is said to have the spectral decomposition property (SDP) if, for every finite open cover $\{G_i\}_{i=0}^n$ of $\mathbf{C}$ (or $\sigma(T)$), where G_0 is a neighborhood of infinity (i.e. its complement G_0^c is compact in $\mathbf{C}$), there exists a system $\{Y_i\}_{i=0}^n$ of invariant subspaces under T satisfying the following conditions:

(I) $X_i \subset D_T$ if G_i $(1 \le i \le n)$ is relatively compact;
(II) $\sigma(T|X_i) \subset G_i$ $(0 \le i \le n)$;
(III) $X = \sum_{i=0}^n X_i$.

The theory based on this property is greatly simplified by the fact [3, Corollary 6.3] that T has the SDP iff it has the two-summand spectral decomposition property that corresponds to $n = 1$. The theory also involves the concept of the spectral manifold $X(T, H) = \{x \in X : \sigma(x, T) \subset H\}$, where $H \subset \mathbf{C}$ and $\sigma(x, T)$ is the local spectrum at $x \in X$, and the concept of the T-bounded spectral maximal space $\Xi(T, F)$ for $F \subset \mathbf{C}$ compact. The T-bounded spectral maximal space $\Xi(T, F)$ is associated to $X(T, F)$ [3, Theorem 4.34] by

$$X(T, F) = \Xi(T, F) \oplus X(T, \varnothing) \quad \text{and} \quad \sigma(T|\Xi(T, F)) = \sigma(T|X(T, F)).$$

The given operator T may enjoy two specific properties:

T is said to have property (β) [1, Definition 8 and 3, Definition 5.5] if, for any sequence $\{f_n : G \to D_T\}$ of analytic functions, the condition $(\lambda - T)f_n(\lambda) \to 0$ (as $n \to \infty$) in the strong topology of X and uniformly on every compact subset of G implies that $f_n(\lambda) \to 0$ in the strong topology of X and uniformly on every compact subset of G.

T is said to have property (κ) [3, Definition 5.4] if T has the single valued extension property and $X(T, F)$ is closed for every closed F.

Property (β) implies property (κ), as follows from [3, Proposition 5.6].

5.2. Lemma. *Suppose that $S \in C(X^*)$. Then S is the dual of a closed and densely defined operator $T \in C_d(X)$ iff $G(S)$ is closed in the weak* topology of $X^* \oplus X^*$ and D_S is total.*

Proof. Only if: Suppose that S is the dual of $T \in C_d(X)$, i.e. $S = T^*$. The equality

$$\nu G(S) = \nu G(T^*) = (G(T))^{\perp}$$

implies $G(S)$ is closed in the weak* topology of $X^* \oplus X^*$. To prove that D_S is total, let $x \in X$ and $\langle x, x^* \rangle = 0$ for all $x^* \in D_S$. Then

$$\langle x, x^* \rangle = 0 = \langle 0, Sx^* \rangle$$

is equivalent to

$$0 \oplus x \in {}^{\perp}(\nu G(s)) = G(T)$$

and hence $x = T(0) = 0$, so D_S is total.

If: Assume that $G(S)$ is closed in the weak* topology of $X^* \oplus X^*$ and D_S is total. Letting $W = {}^{\perp}(\nu G(S))$, one has $W = {}^{\perp}\nu G(S)$. Let $0 \oplus y \in W$. For every $x^* \in D_S$, one has $0 \oplus y \perp (-Sx^*) \oplus x^*$, or equivalently,

$$0 = \langle 0, Sx^* \rangle = \langle y, x^* \rangle \quad \text{for all } x^* \in D_s. \tag{5.1}$$

D_S being total, (5.1) implies that $y = 0$ and hence W is the graph of an operator T. W being closed, T is a closed operator.

To show that T is densely defined, let $x^* \in X^*$ satisfy condition

$$\langle x, x^* \rangle = 0 \quad \text{for all } x \in D_T.$$

Then

$$x \oplus Tx \perp x^* \oplus 0 \quad \text{for all } x \in D_T$$

and hence $x^* \oplus 0 \in (G(T))^{\perp} = W^{\perp} = \nu G(S)$. Therefore $x^* = -S(0) = 0$ and hence T is densely defined. □

5.3. Lemma. *Suppose that $T \in C(X)$ and Y is invariant under T. Then T/Y is closed iff $G(T/Y)$ is topologically isomorphic to $G(T)/G(T|Y)$.*

Proof. Only if: Assume that T/Y is closed. For $x \in D_T$, the following mapping $x \oplus Tx + G(T|Y) \to (x \oplus Y) + (Tx + y)$ is bijective from $G(T)/G(T|Y)$ onto $G(T/Y)$. It follows from the inequalities

$$\begin{aligned} \|x \oplus Tx + G(T|Y)\| &= \inf\{\|x \oplus Tx + y \oplus Ty\| : y \in D_{T|Y}\} \\ &\geq \inf\{\|(x + y_1) \oplus (Tx + y_2)\| : y_1, y_2, \in Y\} \\ &= \|(x + Y) \oplus (Tx + Y)\| \end{aligned}$$

and from the open mapping theorem that $G(T/Y)$ and $G(T)/G(T|Y)$ are topologically isomorphic.

If: Assume that $G(T/Y)$ and $G(T)/G(T|Y)$ are topologically isomorphic. Then $G(T/Y)$ is a Banach space and hence it is closed in $X/Y \oplus X/Y$. Thus T/Y is closed. □

5.4. Lemma. *Given $T \in C_d(X)$, let $Z \subset D_T$ be an invariant subspace under T. Then*

(i) *$Z^\perp$ is invariant under T^*;*

(ii) *$T^*/Z^\perp$ is the dual of $T|Z$ iff $T^*/Z^\perp$ is closed.*

Proof. (i) is evident.

(ii): If $T^*/Z^\perp$ is the dual of $T|Z$ then clearly $T^*/Z^\perp$ is closed. Conversely, assume that $T^*/Z^\perp$ is closed. Then, it follows from Lemma 5.3 that $G(T^*/Z^\perp)$ is topologically isomorphic to $G(T^*)/G(T^*|Z^\perp)$. The following equalities

$$\nu G(T^*) = (G(T))^\perp; \qquad G(T^*|Z^\perp) = G(T^*) \cap (Z^\perp \oplus Z^\perp)$$

imply that both $G(T^*)$ and $G(T^*|Z^\perp)$ are closed in the weak* topology of $X^* \oplus X^*$. Then, it follows easily that $G(T^*/Z^\perp)$ is closed in the weak* topology of $X^*/Z^\perp \oplus X^*/Z^\perp$.

It follows from Lemma 5.2 that D_{T^*} is total and hence $D_{T^*/Z^\perp}$ is total. Quoting again Lemma 5.2, it follows that $T^*/Z^\perp$ is the dual of a densely defined closed operator $U \in C_d(Z)$.

The assumption $Z \subset D_T$ implies that $T|Z$ is bounded. Let $(x^*)\hat{}$ be the equivalence class of $x^* \in X^*$ in $X^*/Z^\perp$. Then, for every $x^* \in D_{T^*}$ and $x \in D_U$, one has

$$\begin{aligned}\langle Tx, x^*\rangle &= \langle x, T^*x^*\rangle = \langle x, (T^*/Z^\perp)(x^*)\hat{}\rangle \\ &= \langle Ux, (x^*)\hat{}\rangle = \langle Ux, x^*\rangle.\end{aligned}$$

Since D_{T^*} is total, (5.2) implies that $Tx = Ux$, for each $x \in D_U$. Since $T|Z$ is bounded and U is a densely defined closed operator, it follows that $U = T|Z$ and hence $T^*/Z^\perp$ is the dual of $T|Z$. □

Now we are in a position to prove our main theorem.

5.5. Theorem. *Given $T \in C_d(X)$, the following assertion are equivalent:*

(i) *T has the SDP;*

(ii) *for every pair of open disks G, H with $\overline{G} \subset H$, there exist invariant subspaces X_G and X_H such that*

$$X = X_G + X_H; \qquad X_H \subset D_T; \tag{5.3}$$

$$\sigma(T|X_H) \subset H \quad \text{and} \quad \sigma(T|X_G) \subset G^c; \tag{5.4}$$

(iii) *for every pair of open disks G, H with $\overline{G} \subset H$, there exist invariant subspaces Y, Z such that*

(a) *$\sigma(T|Y) \subset G^c$; T/Y is bounded and $\sigma(T/Y) \subset H$;*

(b) $Z \subset D_T$, $\sigma(T|Z) \subset H$, T/Z *is closed and* $\sigma(T/Z) \subset G^c$;
(c) $T^*/Z^\perp$ *is closed*;
(iv) *both* T *and* T^* *have property* (β);
(v) T *has property* (β) *and* T^* *has property* (κ).

Proof. The proof will be carried out through the following scheme of implications:

$$\begin{matrix} \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)} \\ \text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \end{matrix} \quad \Rightarrow \text{(v)} \rightarrow \text{(i)}.$$

(i) $\Rightarrow$ (ii) is evident.

(i) $\Rightarrow$ (iii): Given T with the SDP, let G, H be open disks with $\overline{G} \subset H$ and let L be an open set satisfying inclusions $\overline{G} \subset L \subset \overline{L} \subset H$. For $Y = X(T, G^c)$ and $Z = \overline{\Xi(T, L)}$, we have $X = Y + Z$. Then, in view of [3, Proposition 3.4 and Corollary 3.3], conditions (a) and (b) of (iii) are satisfied. Furthermore, it follows from (i) and [3, Theorem 9.8 (II,ii)], that

$$Z^\perp = X^*(T^*, L^c).$$

Consequently, (iii,c) follows from [3, Proposition 3.4].

(ii) $\Rightarrow$ (iv): Let G and H be a pair of open disks with $\overline{G} \subset H$. There exists invariant subspaces X_G and X_H satisfying conditions (5.3) and (5.4). It follows from [3, Proposition 3.4] that T/X_G is bounded and

$$\sigma(T/X_G) \subset \sigma(T|X_H) \cup \sigma(T|X_G \cap X_H) \subset H.$$

Then [3, Theorem 5.8] implies that T has property (β).

To show that T^* has property (β), let $\{f_n^*\}$ be a sequence of D_{T^*}-valued analytic functions defined on an open set $G \subset \mathbf{C}$ such that

$$(\lambda - T^*) f_n^*(\lambda) \to 0 \quad (\text{as } n \to \infty)$$

uniformly on every compact subset of G in the strong topology of X^*. Without loss of generality, we may suppose that $G = \{\lambda: |\lambda| < r\}$ for some $r > 0$ and that $K \subset G$ is compact. Let G_0 and H_0 be open disks satisfying inclusions

$$K \subset G_0 \subset \overline{G}_0 \subset H_0 \subset \overline{H}_0 \subset G.$$

Since T has property (β), the subspaces $X(T, G_0^c)$, $\Xi(T, \overline{H}_0)$ are defined. In view of conditions (5.3) and (5.4) applied to the open disks G_0, H_0, one obtains

$$X = X(T, G_0^c) + \Xi(T, \overline{H}_0). \tag{5.5}$$

Since $K \subset \rho(T|X(T, G_0^c))$, for $\lambda \in K$ and $x \in X(T, G_0^c)$, one has

$$|\langle x, f_n^*(\lambda)\rangle| = |\langle R(\lambda; T|X(T, G_0^c))x, (\lambda - T^*) f_n^*(\lambda)\rangle| \le M_0 \|(\lambda - T^*) f_n^*(\lambda)\| \cdot \|x\|,$$

where $M_0 > 0$ is a constant independent of $\lambda \in K$. Then for every $\varepsilon > 0$, there exists $N_0 > 0$ such that

$$|\langle x, f_n^*(\lambda)\rangle| \le \varepsilon \|x\|, \quad \text{for all } \lambda \in K \text{ as } n > N_0. \tag{5.6}$$

Let $C_0 = \{\lambda: |\lambda| = r_0\} \subset G$ with $\overline{H}_0$ in the interior of the disk bounded by C_0, for some $r_0 > 0$. Then $C_0 \subset \rho(T|\Xi(T, \overline{H}_0))$ and hence for $\lambda \in C_0$ and $x \in \Xi(T, \overline{H}_0)$ one has

$$|\langle x, f_n^*(\lambda)\rangle| = |\langle R(\lambda; T|\Xi(T, \overline{H}_0))x, (\lambda - T^*)f_n^*(\lambda)\rangle| \le M_1\|(\lambda - T^*)f_n^*(\lambda)\|,$$

where $M_1 > 0$ is a constant independent of $\lambda \in C_0$. Then there is N_1 such that

$$|\langle x, f_n^*(\lambda)\rangle| \le \varepsilon \frac{\operatorname{dist}(K, C_0)}{r_0}\|x\| \quad \text{for all } \lambda \in C_0 \text{ as } n > N_1.$$

It follows from the Cauchy integral formula that

$$|\langle x, f_n^*(\lambda)\rangle| \le \frac{1}{2\pi}\int_{|\xi|=r_0} \frac{|\langle x, f_n^*(\lambda)\rangle|}{|\xi - \lambda|}|d\xi| \le \varepsilon\|x\|, \tag{5.7}$$

for all $\lambda \in K$ as $n > N_1$.

The decomposition (5.5) and the inequalities (5.6), (5.7) imply that there is a constant $M > 0$ such that

$$|\langle x, f_n^*(\lambda)\rangle| \le \varepsilon M\|x\| \quad \text{for all } x \in X,\ \lambda \in K \text{ as } n > \max\{N_0, N_1\}.$$

Thus if follows that $\{f_n^*(\lambda)\}$ converges to zero uniformly on K in the strong topology of X^* and hence T^* has property (β).

(iii) $\Rightarrow$ (iv): Condition (iii,a) and [3, Theorem 5.8] imply that T has property (β). By Lemma 5.4, $Z^* = Z^\perp$ is invariant under T^* and then

$$\sigma(T^*|Z^\perp) = \sigma(T/Z) \subset G^c.$$

Again, by Lemma 5.4, $T^*/Z^\perp$ is bounded and hence so is $T|Z$. We have

$$\sigma(T^*/Z^\perp) = \sigma(T|Z) \subset H.$$

Thus [3, Theorem 5.8] applies again and states that T^* has property (β).

(iv) $\Rightarrow$ (v) is evident.

(v) $\Rightarrow$ (i): Let $\{G_0, G_1\}$ be an open cover of $\mathbf{C}$, where G_0 is a neighborhood of infinity and G_1 is relatively compact. Let U_1, U_2 be a couple of Cauchy domains with U_1 bounded, U_2 unbounded such that $U_2 = (\overline{U}_1)^c$. Furthermore, we request that U_2 verify inclusions

$$G_1^c \subset U_2 \subset \overline{U}_2 \subset G_0.$$

Next, we define the linear manifolds N and M as in §4. We claim that the following inclusions hold:

$$\text{(a)}\quad N \subset \overline{X(T, G_0)}, \qquad \text{(b)}\quad \overline{M}^w \subset \Xi^*(T^*, \overline{G}_1). \tag{5.8}$$

To prove (5.8a), let $x \in N$. For $n = 1, 2, 3, \ldots$ choose $f_n \in D_H$ such that $\|x - Hf_n\| < 1/n$. Since T has property (β), $\{f_n\}$ converges uniformly on compact sets in U_1. Put $f(\lambda) = \lim_{n\to\infty} f_n(\lambda)$, for $\lambda \in U_1$. Then $f(\lambda) \in D_T$ and $(\lambda - T)f(\lambda) = x$, $\lambda \in U_1$. Consequently,

$$\sigma(x, T) \subset U_1^c = \overline{U}_2 \subset G_0$$

and (5.8a) follows.

To prove (5.8b), let $x^* \in M$. There exists $g \in D_{H^*}$ such that $H^* g = 0$ and $\tau g = x^*$, or equivalently,

$$(\lambda - T^*)g(\lambda) = \tau g = x^*, \qquad \lambda \in U_2.$$

Thus it follows that

$$\sigma(x^*, T^*) \subset U_2^c \subset \overline{G}_1$$

and hence $x^* \in X^*(T^*, \overline{G}_1)$. Since $g(\lambda) \in V^*$ implies $\lim_{\lambda \to \infty} \|g(\lambda)\| = 0$, it follows from [3, Lemma 5.11] that $x^* \in \Xi^*(T^*, \overline{G}_1)$. Therefore, $M \subset \Xi^*(T^*, \overline{G}_1)$. Now [3, Theorem 9.4] implies that $\Xi^*(T^*, \overline{G}_1)$ is weak* closed and hence $\overline{M}^w \subset \Xi^*(T^*, \overline{G}_1)$. Now (5.8) and Theorem 4.2 imply

$$(5.9) \qquad (X(T, G_0))^\perp \subset N^\perp = \overline{M}^w \subset \Xi^*(T^*, \overline{G}_1).$$

With G_0 fixed, we may choose a sequence of open sets $\{G_n\}$ such that $\bigcap_{n=1}^\infty \overline{G}_n = G_0^c = F_0$ and $\{G_0, G_n\}$ covers $\mathbf{C}$ for every n. Then (5.9) implies that

$$(X(T, G_0))^\perp \subset \Xi^*(T^*, \overline{G}_n) \quad \text{for every } n.$$

Consequently,

$$(5.10) \qquad (X(T, \overline{G}_0))^\perp \subset \bigcap_{n=1}^{\infty} \Xi^*(T^*, \overline{G}_n) = \Xi^*(T^*, F_0).$$

Combining (5.10) with the evident inclusion $(X(T, G_0))^\perp \supset \Xi^*(T^*, F_0)$, one finds

$$(5.11) \qquad (X(T, G_0))^\perp = \Xi^*(T^*, F_0).$$

Since $\Xi^*(T^*, F_0)$ is invariant under T^*, (5.11) implies that $\overline{X(T, G_0)}$ is invariant under T. In fact, for every $x \in X(T, \overline{G}_0) \cap D_T$ and $x^* \in \Xi^*(T^*, F_0)$, one has $\langle Tx, x^* \rangle = \langle x, T^* x^* \rangle = 0$ so that $\overline{X(T, G_0)}$ is invariant under T. Furthermore, we shall show that

$$(5.12) \qquad \sigma(T | \overline{X(T, G_0)}) \subset \overline{G}_0.$$

Let $x \in \overline{X(T, G_0)}$ and choose a sequence $\{x_n\} \subset X(T, G_0)$ such that $x_n \to x$. Let $x_n(\cdot)$ denote the local resolvent of T at x_n. By property (β), the convergence

$$(\lambda - T)x_n(\lambda) = x_n \to x, \qquad \lambda \in (\overline{G}_0)^c$$

implies $x_n(\lambda) \to f(\lambda)$ and $(\lambda - T)f(\lambda) = x$.

Therefore $\sigma(x, T) \subset \overline{G}_0$. On the other hand, for every $\lambda \in (\overline{G}_0)^c$, we have

$$\sigma(x_n(\lambda), T) = \sigma(x_n, T) \subset G_0,$$

so $x_n(\lambda) \in X(T, G_0)$ and hence $x(\lambda) \in \overline{X(T, G_0)}$ for $\lambda \in (\overline{G}_0)^c$. Then, by a known property [5, see also 3, Proposition 2.7], inclusion (5.12) follows.

Now we are in a position to show that T has the SDP. Let $\{G_0, G_1\}$ be an open cover of $\mathbf{C}$ with G_0 a neighborhood of infinity and G_1 relatively compact.

Let H_0 be another open neighborhood of infinity such that $\overline{G}_1 \cap \overline{H}_0 = \varnothing$ and $\overline{H}_0 \subset G_0$. Then $\widetilde{G}_0 = G_1 \cup H_0$ is a neighborhood of infinity and in virtue of (5.11) one writes

$$X(T, \widetilde{G}_0)^{\perp} = \Xi^*(T^*, \widetilde{F}_0),$$

where $\widetilde{F}_0 = (\widetilde{G}_0)^c$ and both $\Xi^*(T^*, F_0)$, $\Xi^*(T^*, \widetilde{F}_0)$ are closed in the weak* topology of X^*. Similarly, $\Xi^*(T^*, F_0 \cup \widetilde{F}_0)$ is closed in the weak* topology. Since $F_0 \cap \widetilde{F}_0 = \varnothing$ $(F_0 = G_0^c)$, we have

$$\Xi^*(T^*, F_0 \cup \widetilde{F}_0) = \Xi^*(T^*, F_0) \oplus \Xi^*(T^*, \widetilde{F}_0). \tag{5.13}$$

Set $Z^* = \Xi^*(T^*, F_0 \cup \widetilde{F}_0)$.

Let $x \in X$, $x^* \in Z^*$ and denote by x_0^* the projection of x^* onto $\Xi^*(T^*, F_0)$, in conjunction with (5.13). The linear functional x_0 on Z^*, defined by

$$\langle x_0, x^* \rangle = \langle x, x_0^* \rangle \tag{5.14}$$

is continuous in the weak* topology. Use the Hahn-Banach theorem on locally convex spaces to extend x_0 to a linear functional on X^*, that is continuous in the weak* topology. Therefore $x_0 \in X$. Since the projection x_0^* of $x^* \in \Xi^*(T^*, \widetilde{F}_0)$ onto $\Xi^*(T^*, F_0)$ is zero, it follows from (5.14) that $\langle x_0, x^* \rangle = 0$ for $x^* \in \Xi^*(T^*, \widetilde{F}_0)$. Thus, $x_0 \in {}^{\perp}\Xi^*(T^*, \widetilde{F}_0) = \overline{X(T, \widetilde{G}_0)}$. Put $x_1 = x - x_0$ and for $x^* \in \Xi^*(T^*, F_0)$, use (5.14) to obtain $\langle x_1, x^* \rangle = 0$. Then $x_1 \in {}^{\perp}\Xi^*(T^*, F_0) = \overline{X(T, G_0)}$. Since $x \in X$ is arbitrary, the representation $x = x_0 + x_1$ with $x_0 \in \overline{X(T, \widetilde{G}_0)}$, $x_1 \in \overline{X(T, G_0)}$ implies

$$X = \overline{X(T, G_0)} + \overline{X(T, \widetilde{G}_0)}. \tag{5.15}$$

As regarding $\overline{X(T, \widetilde{G}_0)}$, it follows from (5.12) that

$$\sigma(T | \overline{X(T, \widetilde{G}_0)}) \subset \overline{\widetilde{G}}_0 = \overline{G}_1 \cup \overline{H}_0.$$

Having $\overline{G}_1 \cap \overline{H}_0 = \varnothing$ and G_1 relatively compact, the functional calculus for closed operators produces the following decomposition

$$\text{(a)} \quad \overline{X(T, \widetilde{G}_0)} = Y_1 \oplus Y_2, \qquad \text{(b)} \quad Y_1 \subset D_T; \tag{5.16}$$

$$\text{(a)} \quad \sigma(T | Y_1) \subset \overline{G}_1, \qquad \text{(b)} \quad \sigma(T | Y_2) \subset \overline{H}_0. \tag{5.17}$$

Since $\overline{H}_0 \subset G_0$, $Y_2 \subset \overline{X(T, G_0)}$, (5.15) and (5.16) imply

$$X = Y_1 + \overline{X(T, G_0)}. \tag{5.18}$$

In view of (5.16b), (5.12), (5.17a) and (5.18), T has the SDP.

Remark. A more restrictive version of property (β) is used in [6, Lemma 4.6]. Given $T \in C(X)$, a function $f\colon G \to D_T$ defined on an open subset G of the compactified complex plane $\mathbf{C}_\infty$, is said to be T-analytic if both f and Tf are analytic on G. T has property (β), in this stronger version, if for any sequence of T-analytic functions $\{f_n : G \to D_T\}$, the condition $(\lambda - T)f_n(\lambda) \to 0$ (as $n \to \infty$) in the strong topology of X and uniformly on every compact subset of G implies that $f_n(\lambda) \to 0$ in the strong topology of X and uniformly on every compact subset of G.

It follows from the definition of the operator H and Lemma 2.2 in §2 that both $Tf(\mu)$ and $T^* g(\lambda)$ are analytic. Consequently, Theorem 5.5 holds if we use the above-mentioned stronger version of property (β) in (iv) and (v).

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